P. Malbéqui
(Onera)

E-mail : patrice.malbequi@onera.fr

DOI : 10.12762/2014.AL07-09

Capabilities of the High-Order Parabolic Equation to Predict Sound Propagation in Boundary and Shear Layers

The so-called parabolic equation (PE) has proved its capability to deal with the long range sound propagation as an alternative to the ray model. It was shown that the High-Order Parabolic Equation (HOPE), based on a Padé expansion, significantly increases the aperture angle of propagation, compared to the standard PE and the wide-angle PE. As a result, for the in-duct propagation it allows us an accurate prediction close to the cut-off frequency. This paper concerns the propagation using the HOPE in heterogeneous flows, including a boundary layer above a hard wall and in shear layers. The thickness of the boundary layer is some dozens of centimeters while, outside of it, the Mach number can reach 0.5. The flow effects are investigated showing the refraction effects at a propagation distance of 30 meters, up to a few kilohertz. Significant discontinuities in the directivity patterns occur in the shear layer. Comparisons with the Euler solution are considered, including configurations beyond the theoretical limits of the HOPE.

Introduction

The propagation of waves in a heterogeneous medium is a problem widely encountered in aero-acoustics for the prediction of the noise radiated by an aircraft. For instance, the fan noise radiation from the aero-engine represents a significant acoustical source during take-off. The optimization of absorbent liners on the nacelle walls and the design of the nacelle shape are efficient means to reduce the noise. The modeling of this configuration includes the liner absorption and the flow effects. To deal with such a problem in the very large frequency range of interest, from a few dozen Hertz to few a kilo-Hertz (i.e., a reduced wave number $k_0 a$ varying from 1 to 100, where k_0 is the wave-number and a is the duct radius), several complementary methods are required. Recent progress in Computational Aero-Acoustics (CAA) allows us to analyze wave propagation in a complex medium for realistic configurations, solving the Euler's equations with a high-order finite difference scheme. Among the CAA techniques, the latter are the less restrictive regarding the flow properties. They can handle a rotational flow such as the one that evolves in an exhaust jet. However, at present time, they are still limited to the low frequency range, due to the CPU time requirement. The Boundary Element Method (BEM) is applicable over almost the entire frequency range of interest, assuming a homogeneous flow in the radiated far-field [12]. As an approximation to the BEM, the fast multi-pole method allows realistic configurations in aero-acoustics to be modeled, including the shielding effects [8]. The coupled FEM/BEM and the FEM/Infinite elements can be used to take into account a potential flow [18], [1] in the low and mid frequency range. The ray-model is widely used in the high-frequency approximation. In addition to propagation in a heterogeneous medium, it has been applied for predicting the propagation of broadband fan-noise and to analyze the effect of scarfing on the radiation from an intake [11]. However, taking into account both an obstacle and a non-uniform flow is difficult, especially integrating the ray equations together with the implementation of the scattering effects and creeping waves on the obstacle surface. There is a need for the modeling of wave propagation in the high frequency range. This paper is aimed at illustrating that the High-Order Parabolic Equation (HOPE) is a candidate for computing propagation in the mid and high frequency range.

The Parabolic Equation (PE) is used in many domains, such as electromagnetic propagation, seismic waves, underwater acoustics and long range sound propagation in the atmosphere [4]. Numerical solutions of wave scattering problems in the parabolic approximation are presented in [5], for instance, the scattering of plane waves by a refraction index inhomogeneity with an elliptical cross–section and the scattering of plane waves by a viscous core vortex, which are of interest for problems such as propagation in a shear layer of a free jet. These situations may occur for internal noise radiating from the jet engine pipe, or in anechoic open wind tunnel facilities used to simulate the forward flight effects. The solutions are in good agreement with the Born approximation, including on the right hand-side an index of refraction and its derivative in the incidence direction.

This paper describes sound propagation in the presence of refractions. It starts with a brief derivation of the High-Order Parabolic Equation. Its application to propagation in boundary and shear layers is then presented. The results are compared to reference solutions.

Theoretical background

The PE is derived from the elliptic wave equation in a heterogeneous medium. Neglecting the back-scattered field, the PE uses a marching method to efficiently solve the forward propagating field. The PE involves an approximation of the square root operator, including a second order derivative of the pressure field.

The PE is currently used for outdoor propagation and underwater acoustics. Few applications are also devoted to duct acoustics, to predict the noise radiated from an aeroengine nacelle. In the past, Baumeister [3] proposed the numerical spatial marching techniques for in-duct propagation, where he examined the stability problem of the finite difference technique. Dougherty [9] developed a method based on the parabolic approximation to the convected Helmholtz equation in an orthogonal curvilinear coordinate system, to analyze the effects on sound propagation in non-uniform, softwall ducts. Nark et *al.* [15] coupled the duct propagation in the parabolic approximation with the radiation technique in the far-field based on the Ffwocs Williams-Hawkings approach.

In the past, Onera developed the PARABOLE code based on the Wide-Angle PE (WAPE) for outdoor propagation, taking into account the temperature and the wind gradients, the sound absorption by the ground and the soil topography [13]. The HOPE is based on the Padé expansion, using a high order to approximate the square root operator. The theoretical background of the HOPE was analyzed by Bamberger et *al.* [2]. It has been used by Collins [7] for underwater acoustics in presence of an elastic bottom and by Malbéqui [14] for duct acoustics, to take into account scattering effects at very large angles, allowing the propagation close to the cut-off frequency to be predicted accurately. This latter work has also shown the interest and the capabilities of the PE for applications on the aeroacoustics problems and has motivated this study dealing with the propagation in heterogeneous flows.

Wave equation in a heterogeneous medium

It is not straightforward to appreciate the validity and the limitations of the PE. Making use of reference cases and of increasing progress in CAA to solve the Euler's equations, it is convenient to examine its validity domain.

Starting from the Lighthill equations and assuming a medium characterized by velocity fluctuations much smaller than the sound celerity $\left(u / c_0 << 1\right)$, temperature fluctuations much smaller than the ambient temperature $\left(T' / T_0 << 1\right)$ and a low mean flow Mach number $\left(M_0 = u_0 / c_0 << 1\right)$, the propagation of a weak acoustic field in the medium may be described by the following wave equation [5]:

$$\Delta p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial}{\partial x_i}\left(\frac{T'}{T_0}\frac{\partial p}{\partial x_i}\right) - 2\rho_0 \frac{\partial^2 u_i v_j}{\partial x_i \partial x_j} \quad (1)$$

where T_0, ρ_0, c_0 are the mean temperature, density and sound velocity, T' and u_i represent the temperature and velocity fluctuations in the medium, and p and v_j are the perturbations associated to the acoustic field.

In the case of the propagation of a monochromatic sound field $(k_0 = \omega / c_0$ with $a\ exp^{(-i\omega t)}$ time dependence) in an almost time independent medium, Eqn. (1) is transformed into an elliptical wave equation.

Parabolic equation

We consider the sound propagation in two-dimensional cylindrical coordinates (r, z), where r is the range and z is the altitude. For compactness, the linear partial differential $[P]$ and $[Q]$ operators are introduced:

$$[P] = \frac{\partial}{\partial r} \quad (2a)$$

$$[Q] = [1+X]^{1/2} \quad (2b)$$

$$[X] = \left[N^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] \quad (2c)$$

Assuming small variations of the index of refraction $N = c_0 / (c+u_r)$ along the main direction of propagation , where u_r is the horizontal component of the flow velocity, after a certain amount of algebra, the elliptical wave equation can be factored into forward and backward propagating components :

$$[P + ik_0 Q][P - ik_0 Q]\, p = 0 \quad (3)$$

The paraxial approximation consists in neglecting the backward field. The pressure field p is written in the form $p(r,z) = \psi(r,z)\exp(ik_0 r)$, using a plane wave assumption, $\exp(ik_0 r)$ and an envelope function with slow variations with respect to the phase of the wave, $\psi(r, z)$. Thus, equation (3) can be transformed into :

$$[P]\psi = ik_0 [Q-1]\psi \quad (4)$$

Finally, to obtain a parabolic equation, the square root operator Q is approximated. The simpler approximation, based on the Taylor expansion of the first order in X, provides the standard PE [10]. The High-Order Parabolic Equation (HOPE) is derived, using an approximation of Q with a Padé expansion of order n :

$$[Q] = \left[1 + \sum_{j=1}^{n} \frac{a_{j,n} X}{1 + b_{j,n} X}\right] + O(X^{2n+1}) \quad (5)$$

When n equals 1, the so-called wide-angle parabolic Equation proposed by Claerbout is obtained [6].

Numerical method

The PE is solved using a finite difference technique and a Crank-Nicholson scheme. We consider Ψ^l_m the field at point $(l\Delta r, m\Delta z)$, where Δr and Δz denote the steps with respect to r and z. The second derivative $\partial^2 / \partial z^2$ is estimated with a central difference of order $O(\Delta z^2)$. At each marching step, from $l\Delta z$ to $(l+1)\Delta r$, the discretization of the PE leads to a linear tri-diagonal system. For its numerical implementation, the HOPE associated to the Padé expansion of order n is transformed into a set of n successive PEs. For compactness, we set :

$$L_{j,n} = \frac{a_{j,n}X}{1+b_{j,n}X} \tag{6}$$

and the HOPE becomes

$$[P]\psi = ik_0\left[\sum_{j=1}^{n} L_{j,n}\right]\psi \tag{7}$$

Applying the Cranck-Nicholson finite difference scheme on the P operator, Eqn. (7) can be transformed into a set of n successive PEs, using the alternating directions technique. The j^{th} PE is written as:

$$\left[1-ik_0\frac{\Delta r}{2}\sum_{j=1}^{n} L_{j,n}\right]\psi^{l+j/n} = \left[1+ik_0\frac{\Delta r}{2}\sum_{j=1}^{n} L_{j,n}\right]\psi^{l+(j-1)/n} \tag{8}$$

The first PE, when $j = 1$, makes use of the known field y^l at range r, and the last one, when $j = n$, provides the expected field ψ^{l+1} at the range $r + \Delta r$.

Boundary conditions

For the numerical implementation of the PE, a Gaussian function is generally used as the starting field, at $r = r_0$. This reduces the aperture angle of the field, in agreement with the paraxial approximation. Using the HOPE, allowing a larger aperture angle of propagation, a spherical radiation from a monopole source can be used (in presence of the image source for the configuration of the boundary layer above a wall).

Two techniques are available to avoid artificial numerical reflections in the upper part of the mesh, at $z = z_{Max}$: to fix the wave impedance of the field on the boundary, or to mesh additional absorbing layers. The first one has been successfully applied for the long range sound propagation in the atmosphere. The wave impedance of the acoustics field, at $z = z_{Max}$, is obtained from the pressure and acoustical velocity, approximated from the solution of spherical waves in a homogeneous medium. In comparison with the absorbing layers, this reduces the mesh size and the CPU time, but for the configurations of interest in this paper with significant gradients it is not efficient. The second technique, with a sponge domain, is applied. Its thickness corresponds to about 50 % of the domain where the sound field is computed. In practice, in the absorbing layers, we add an imaginary part $ig(z)$ to the real wave-number k_0, where $g(z)$ is proportional to z^2, introducing a damping of the field according to a law $exp(-g(z))$.

Using the PE, no boundary condition is required at the end of the mesh, $r = r_{Max}$, avoiding here the tricky problem of the implementation of a non-reflective boundary condition.

Propagation in a boundary layer

For the Euler solution, the code Sabrina developed by Onera including a high-order finite difference scheme is used [17]. The mesh is regular in the domain of interest and stretches in a sponge region to avoid artificial numerical reflections on the boundaries. The solution is time dependent, with a time step Δt of 6.10 $^{-5}$ s. A few millions of grid points are required to simulate the propagation up to 1 kHz, in a range propagation of 30 meters, corresponding to about one hundred acoustical wavelengths, and an altitude of 15 meters.

For the PE solution, the spatial steps in both vertical and horizontal directions are of about a tenth of the wavelength, and the Padé number is 5.

Figures 1, 2, 4 and 5 (next page) compare the PE and the Euler solutions, for various flow conditions. The pressure is plotted in a vertical plane $(0, r, z)$ in decibels and referenced to 0 dB at the distance of 1 m from the point source. It must be noted that, for the Euler method, the point source in the fluid is approximated by a normalized Gaussian-distribution [16], while for the PE a starting field modeling is considered. As a result, level differences are observed very close to the source between the PE and Euler solutions.

The PE and the Euler solutions are plotted in figures 1 and 2 for a Mach number 0.01 and 0.5, in the case of a homogeneous flow. The pressure is plotted in a vertical plane $(0, r, z)$ in decibels. A quite good agreement is shown, including the expected interference pattern between direct and reflected fields (a good agreement is also found with the analytical solution derived from the direct field and reflected fields).

For the sound propagation in the boundary layer, the Mach number profile is defined by : $M(z) = M_0 z/(z+\alpha)$, where M_0 represents the Mach number limit when z goes to infinity and M_0/α is the slope of the profile at $z = 0$ (figure 3).

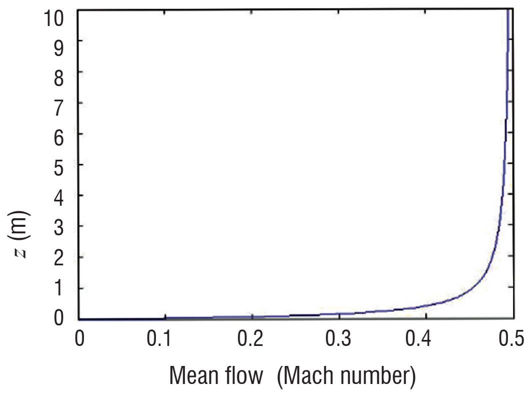
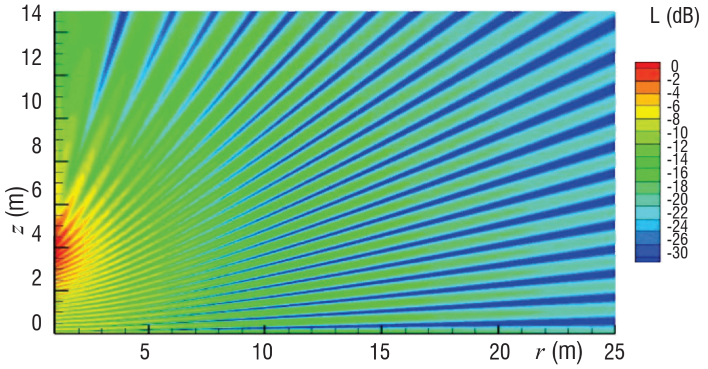


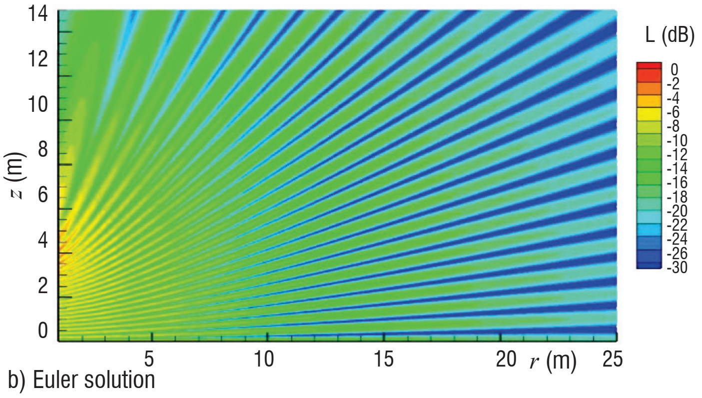
Figure 3 - Profile of the boundary layer above a wall : $M_0 = 0.5$, $\alpha = 0.1$.

Figure 4 shows the directivity pattern in presence of the boundary layer $(M_0 = 0.5, \alpha = 0.1)$. The point source is located at 4 m above the wall, where the gradient of the flow velocity is weaker, so no significant difference is observed between the uniform and the heterogeneous configurations. However, as expected with refraction, the energy tends to be bent towards the wall. A satisfactory agreement between the two solutions is obtained, while the flow velocity and its fluctuations are beyond the theoretical PE assumptions.

Figure 5 illustrates the comparisons at 2 kHz in presence of the boundary layer, where a satisfactory agreement between the Euler and the PE solutions is found.

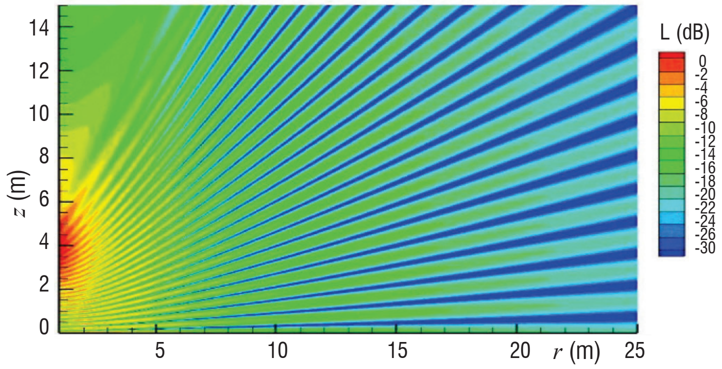


a) Parabolic solution

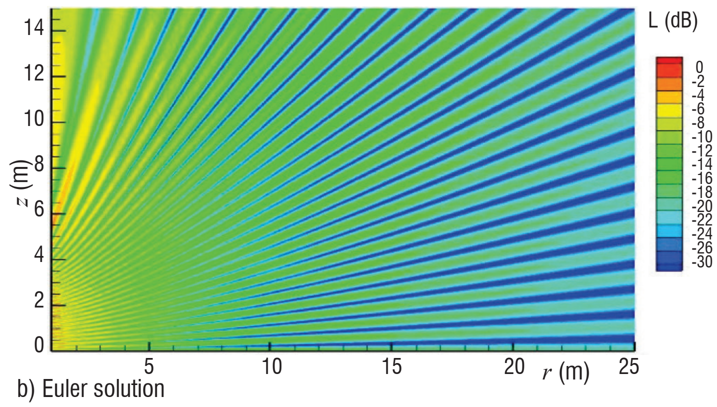


b) Euler solution

Figure 1 - Propagation above a rigid wall, in a uniform flow, $M = 0.01$, f = 1 kHz

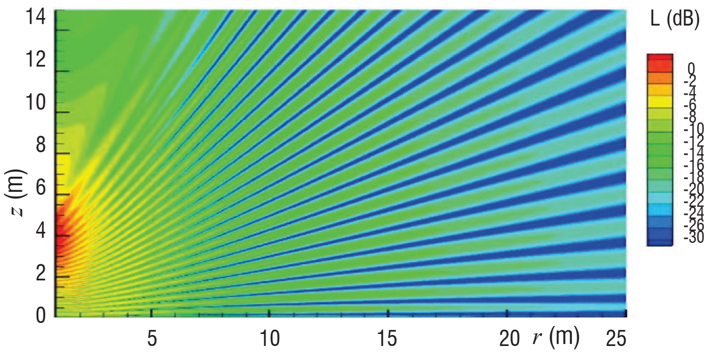


a) Parabolic solution

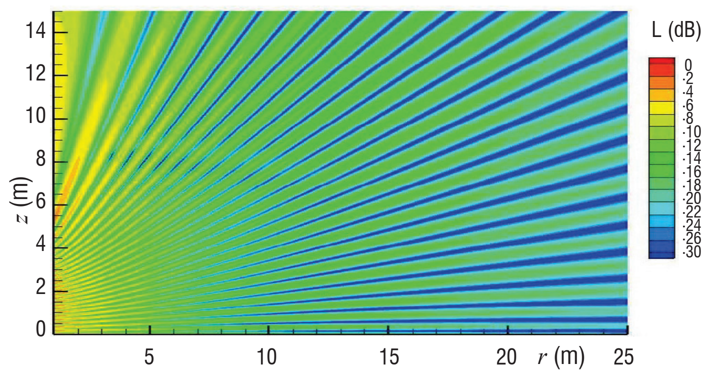


b) Euler solution

Figure 2 - Propagation above a rigid wall in a uniform flow, $M = 0.5$, f = 1 kHz

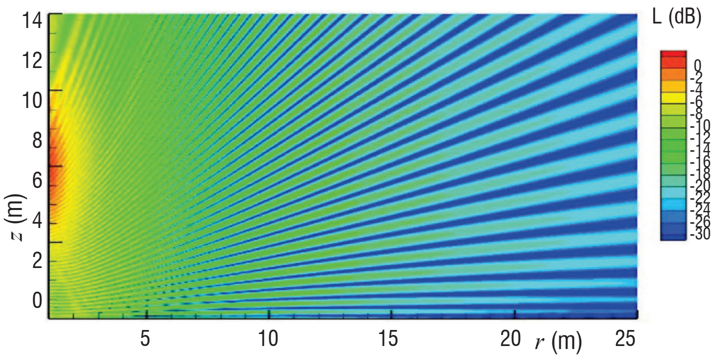


a) Parabolic solution

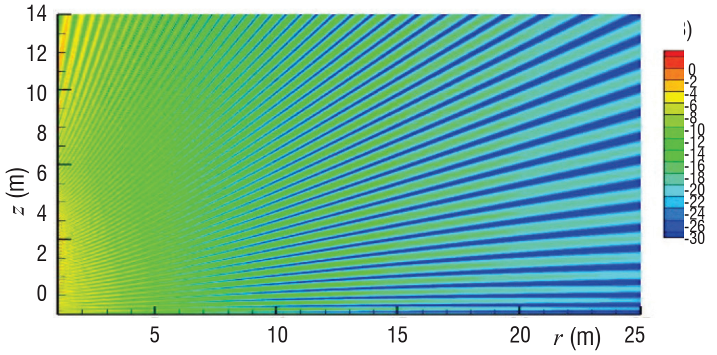


b) Euler solution

Figure 4 - Propagation above a rigid wall in a boundary layer $M_0 = 0.5$, $\alpha = 0.1$, f = 1 kHz



a) Parabolic solution



b) Euler solution

Figure 5 - Propagation above a rigid wall, in a boundary layer $M_0 = 0.5$, $\alpha = 0.1$, f = 2 kHz

Propagation in a shear layer

For the propagation in the shear layer, the Mach number profile is defined by :

$$M(z) = M_0 - M_0^- z/(z - \beta),\ if\ z < 0 \qquad (9a)$$

$$M(z) = M_0 + M_0^+ z/(z + \beta),\ if\ z > 0 \qquad (9b)$$

where M_0^- and M_0^+ represent the Mach number limits when z goes to ± infinity, M_0 equals $(M_0^+ + M_0^-)/2$ and $M_0^{\pm}/\beta$ is the slope of the profile at $z = 0$, see figure 6.

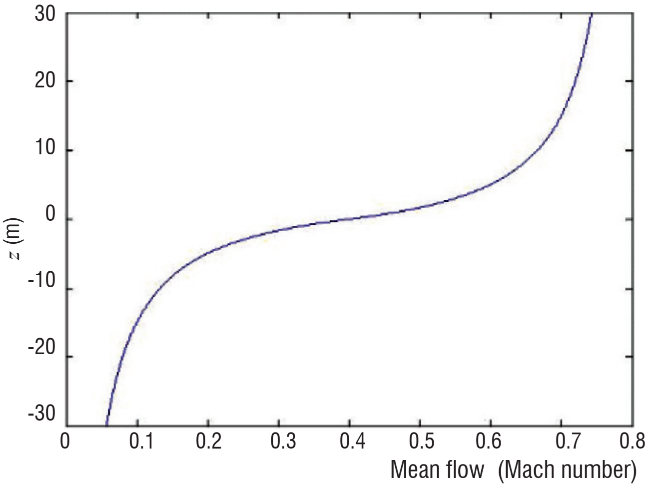
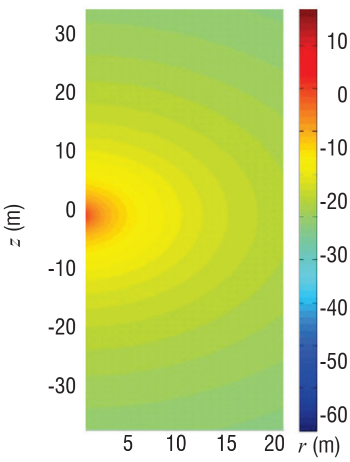
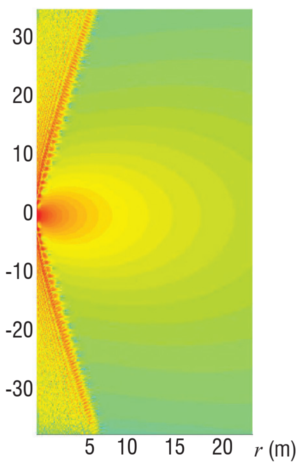


Figure 6 - Profile of the shear layer : $M_0^- = 0.0$, $M_0^+ = 0.8$ and $\beta = 0.5$

We first consider in figure 7 the propagation of a spherical wave in a homogenous medium, with the two Mach numbers 0.1 and 0.8. The directivity pattern shows the HOPE limitation at a very large aperture angle, where a false increase of the pressure level appears from about 75°, when the Mach number reaches 0.8.

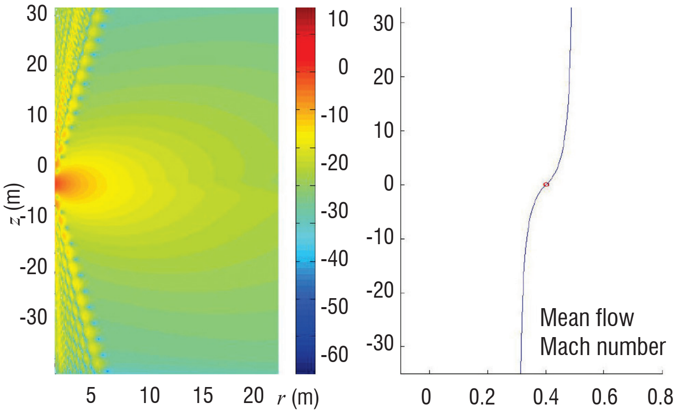


a) $M_0 = 0.1$

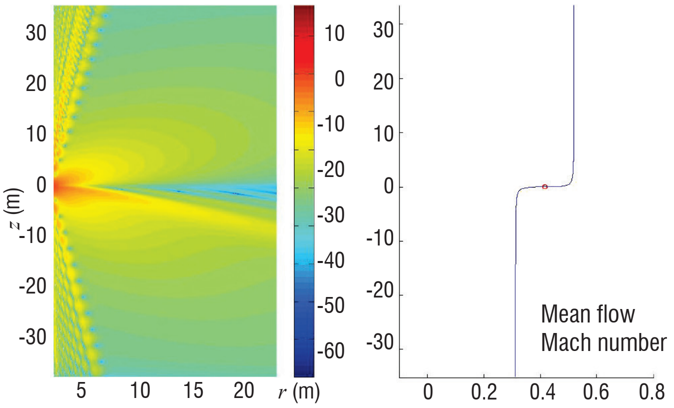


b) $M_0 = 0.8$

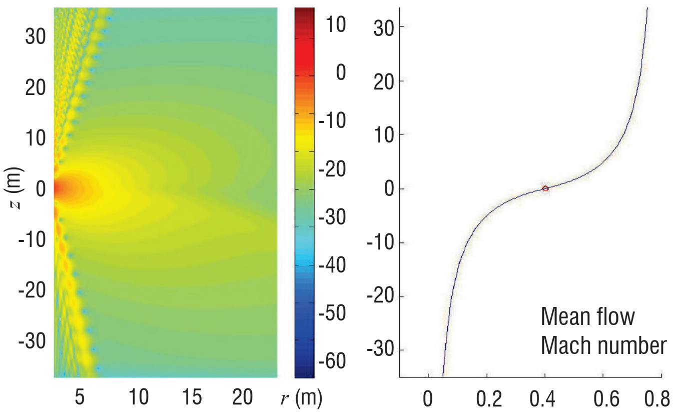
Figure 7 - Propagation in a uniform flow with the Parabolic Equation, f = 1 kHz



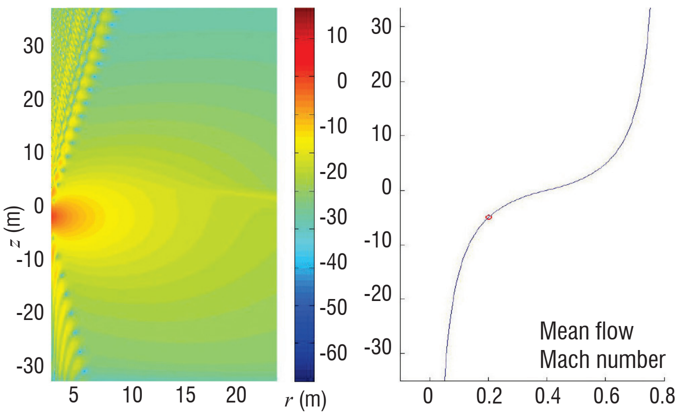
a) $M_0^- = 0.3,\ M_0^+ = 0.5,\ \beta = 5,\ z = 0$



b) $M_0^- = 0.3,\ M_0^+ = 0.5,\ \beta = 0.1,\ z = 0$



c) $M_0^- = 0.0,\ M_0^+ = 0.8,\ \beta = 5,\ z = 0$



d) $M_0^- = 0.0,\ M_0^+ = 0.8,\ \beta = 5,\ z = -5\ m$

Figure 8 - Propagation in a shear flow with the Parabolic Equation, f = 1 kHz

Figure 8 plots, on the left hand-side, the directivity pattern and on the right-hand side the profile of the shear flow (the red point indicates the source location). First, the pattern at the aperture angle of about 75°, where the solution is not valid arises, as observed in figure 7. Figure 8a and figure 8b show the influence of the shape of the shear layer on the sound radiation, when the source is located at the origin. It is found that the increase of the slope variations in the flow modifies the directivity pattern, in particular, it accentuates the differences between the radiation in the upper $(z > 0)$ and the lower half planes $(z < 0)$. Figures 8c and 8d compare the results with the same shear flow profile at two source locations. As expected, when the source gets closer to the origin, the wave refraction becomes more intense.

Figure 9 superimposes the PE solution (constant contour lines) and the Euler solution (scale of color levels) for a shear flow with $M_0^- = 0.4,\ M_0^+ = 0.5, \beta = 5$, when the source is located at $z = -5\ m$. A marked influence of the shear layer appears on the pressure field. A convenient agreement is obtained between the two solutions in the lower half plane, but differences occur on the upper half plane. It must be noted that, in addition to the constitutive equations, the problems solved with the PE and Euler are not exactly the same : as a starting field, the PE solution assumes the radiation of a spherical wave, while in the Euler solution the source radiation modified by the shear layer is no longer a spherical wave.

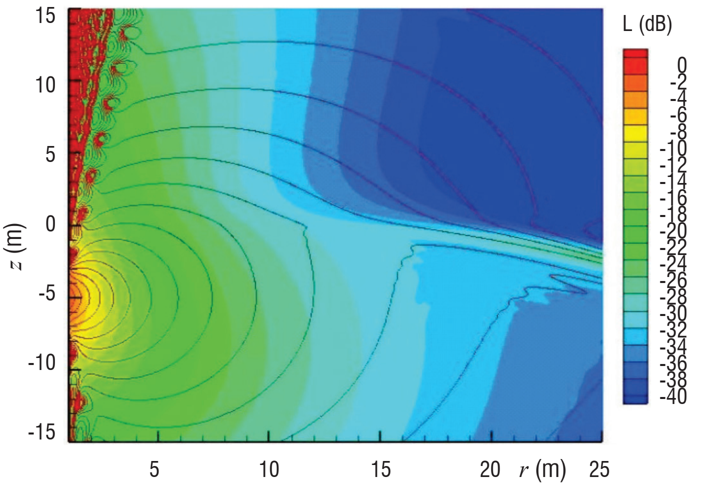


Figure 9 - Propagation in a shear flow, $f = 1$ kHz. Constant contour lines : parabolic solution ; scale of color levels : Euler solution.

Conclusion

The High-Order Parabolic Equation (HOPE) is derived from the wave equation under successive assumptions to predict sound propagation in a heterogeneous flow, within a short CPU time. It is solved with a finite difference scheme, together with the alternating direction method. The HOPE includes an approximation of a square operator using a rational function with a Padé expansion. It is then possible to overcome the limitation of the aperture angle, when using a high Padé order. It seems that neglecting the back-scattered field is not too restrictive, especially in the high frequency range. The PE being derived from the wave equation, its more restrictive assumption concerns the propagation in a potential flow. The PE results are in good agreement with the Euler solution for propagation in a boundary layer above a rigid wall. Differences between the two solutions occur in the case of the shear layer, but the starting fields are not the same. Using the PE, a boundary condition is introduced in the far-field. When solving the Euler equation, a monopolar acoustical source is modeled in the fluid and the modification of its radiation pattern in the near field by the heterogeneous flow is taken into account through the computation.

The PE saves significant computational time compared to the exact solution of the Euler equation, remaining attractive nowadays, especially for applications in the high frequency range, such as the design of a liner on the intake walls of an aeroengine nacelle and, more generally, optimization problems.

Further studies concern the parabolic computations using the Euler solution as starting field, with the application of the HOPE to more realistic geometries using an implementation with a system of curvilinear coordinates. Numerical developments are also required for the extension from 2D to 3D modelization

Acknowledgments

The author acknowledges AIRBUS-France for fruitful discussions and for supporting this study. He would also like to thank D. Simon-Christien, Master Student at Paris Sud XI, for his contribution in the study and D. Mincu, from the Onera CFD and aero-acoustics Department, for his guidance in providing the Euler solution.

Acronyms

BEM (Boundary Element Method)
CAA (Computational Aero-Acoustics)
FEM (Finite Element Method)
HOPE (High Order Parabolic Equation)
PE (Parabolic Equation)
WAPE (Wide-Angle Parabolic Equation)

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AUTHOR

Patrice Malbéqui graduated from TELECOM ParisTech in 1983. He is at present responsible for "Environment and Propagation" in the CFD and Aeroacoustics department at Onera, where he is involved in aircraft noise modelling and in the acoustical environment of launchers at lift-off. As a specialist on sound propagation, he developed a model based on the parabolic equation together with assessment against experimental results. He has some experience in other fields, such as signal processing, duct acoustics, experiments in wind tunnels and liner optimization.