C. Poussot-Vassal, F. Demourant
(Onera)

E-mail: charles.poussot-vassal@onera.fr

Dynamical Medium (Large)-Scale Model Reduction and Interpolation with Application to Aircraft Systems

Although the need for even more accurate system, phenomena and process modeling is required in order to reduce development time and costs, the number of variables linear and non-linear optimization tools can handle is still a practical and theoretical limiting factor. This is especially true in aircraft dynamical performance analysis, monitoring and control design, where dynamical models are accurately designed at varying local flight configurations, in order to handle flexible modes, aerodynamic delays, etc., leading to high-dimensional problems [5]. Although Onera has a well established tradition of proposing complete and efficient tools for optimizing controllers and analyzing dynamic system performances through the use of Linear Fractional Representation (LFR) mathematical objects [2, 15, 22], recent growth in the dimensions of models has led to strong time and computational limitations when using these tools. The aim of this paper is to give an overview of the solutions developed within Onera **to approximate a set of large-scale dynamical models with a parameterized LFR lower order model**, which can be used in place of the original ones to effectively synthesize control laws and achieve performance analysis.

Introduction and main problem

Motivations and challenges in aeronautics

In many areas of engineering (*e.g.* aerospace, automotive, biology, circuits…), dynamical systems are the basic framework used for modeling, control and analysis of a large variety of systems and phenomenon. Due to the growth in the use of dedicated and accurate computer-based modeling design software, numerical simulations have been increasingly used to simulate complex systems or phenomenon and shorten both development time and cost. However, the need for enhanced accuracy of the models has led to an increasing number of variables and resources to be handled at the price of a high and expensive computational cost. Moreover, from the control engineer point of view, modern analysis (*e.g.* LPV H_∞ and H_2 norm computation, μ-analysis…) and synthesis (*e.g.* H_∞, H_2 control…) tools [2, 4, 8, 9, 22] become drastically inefficient for such high-dimensional dynamical systems (see figure 1) [1].

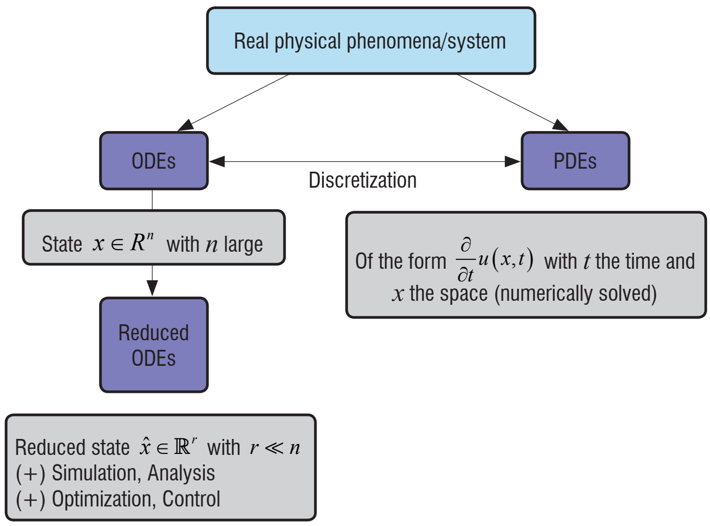


Figure 1 – Illustration of the model reduction paradigm

These remarks are especially true in the flight dynamics domain where aircraft are locally modeled (*i.e.* at each flight and mass point) with high fidelity tools to account for the flexible modes and aeroelastic delays (see *e.g.* [5]).

As an illustration, let us consider figure 2, in which the frequency responses of an industrial longitudinal aeroelastic commercial aircraft are plotted for varying flight points (MACH / Calibrated air speed configurations). The model considered has 3 inputs (the ailerons deflection, the elevator deflections and the vertical wind disturbance) and 3 outputs (the vertical load factor, the bending moments at the tail horizontal plan and at the wing/fuselage positions). This model also includes actuators and a severe von-Karman wind disturbance model. It is worth noting that the entire model has about 300 states and that the system behavior is flight conditions dependent (see also, a very interesting paper on aircraft modeling [5]).

Because of this complexity, the resulting control engineer problem is large and configuration/flight point dependent. More specifically, the high numbers of variables and dynamics lead to two major problems for numerical and control engineers:

• An increase in simulation time and, eventually, in the difficulty of analyzing the model's properties with respect to uncertainties, parameter variations, nonlinearities…;

• The difficulties of controller synthesis. In practice, modern control methods (such as LQG, Robust, MPC…), use the dynamical system model directly and employ optimization methods (*e.g.* descent, LMI, non-smooth…) to synthesize the controllers [2, 22].

Consequently, **the model reduction and interpolation stage**, linking the modeling and the control law design, aims to achieve the following main objectives (see also the very relevant work of Antoulas [1]):

• To **speed up the simulation** in the validation stage, using simpler models, while preserving the most significant system dynamics and properties (*e.g.* frequency response, stability, structure…);

• To **efficiently use the numerical control tools** in order to synthesize controllers in a cleverer manner and thus focus on the controller structure for implementation purposes. Note that most modern control approaches lead to controllers that are considerably more complex than truly needed (mainly because of the initial dimensions of the model);

• To **describe the nonlinear model over the entire parametric domain** (*e.g.* flight point and mass configurations), even if the model is only provided at local configurations.

Mathematical problem definition and Onera approach

Starting from a set of medium (large) scale Linear Time Invariant (LTI) models describing a complex system at frozen configurations, the problem tackled in this paper consists of obtaining a reduced-order Linear Parameter Varying (LPV) model of suitable form, from which a Linear Fractional Representation (LFR) can be built to be used in place of the original LTI models. This objective is formalized in problem 1 (see also [21]).

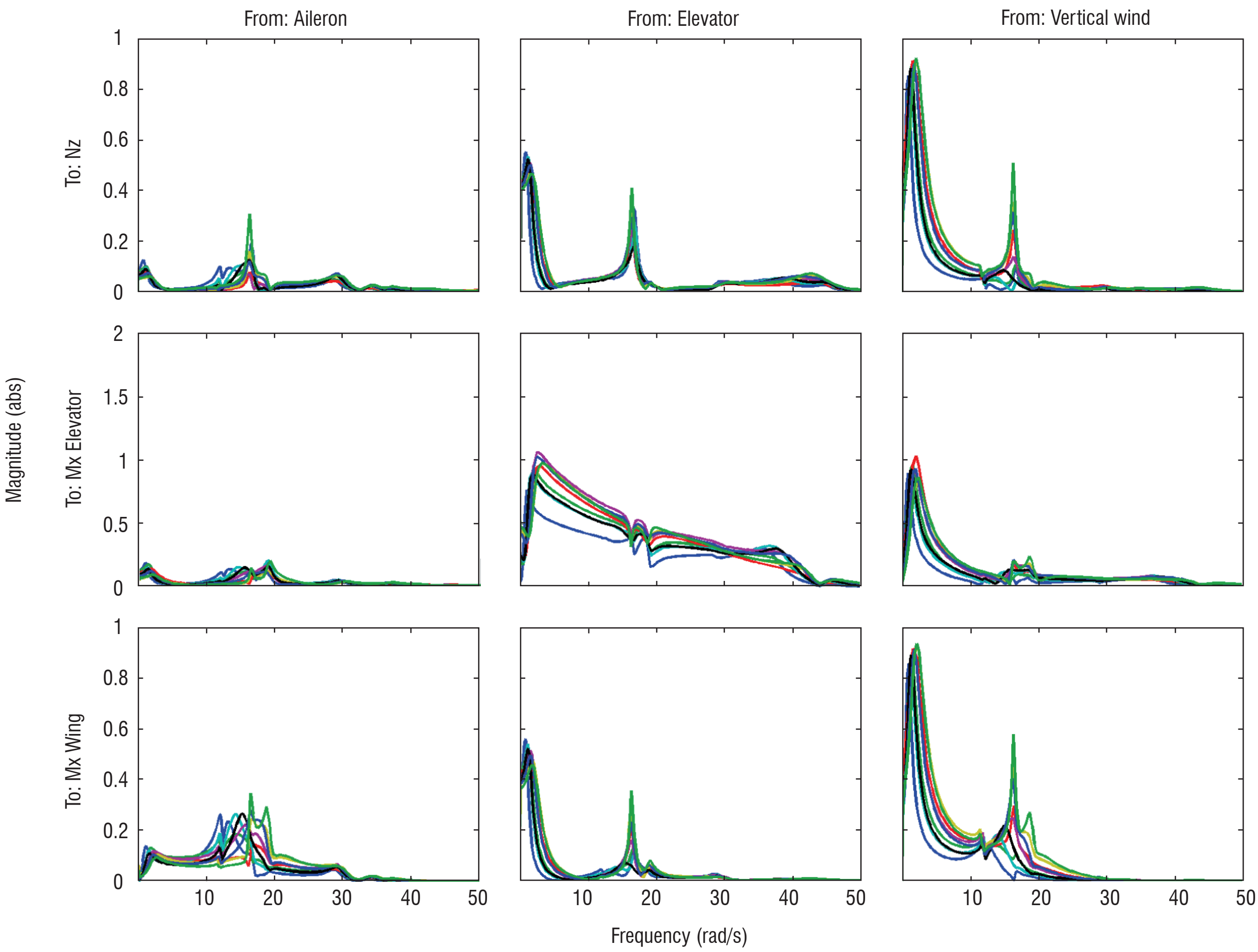


Figure 2 – Frequency response for varying flight configurations (in MACH / Calibrated air speed)

Problem 1 - Multi-models approximation and interpolation

Let us consider a set of N dynamical system models of order n (e.g. defined by n-ODEs), corresponding to different parametric configurations (e.g. flight point, tank filling…), described as follows:

$$G_i : \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u(t) \\ y(t) = C_i x_i(t) \end{cases}, \text{ where } i \in \{1,\ldots,N\}$$

The objective is to find a parameterized model of order $r \ll n$, of the form,

$$\hat{G}(\delta) : \begin{cases} \dot{\hat{x}}(t) = \hat{A}(\delta)\hat{x}(t) + \hat{B}(\delta)u(t) \\ \hat{y}(t) = \hat{C}(\delta)\hat{x}(t) \end{cases}, \text{ where } \delta \text{ varies within a bounded compact set}$$

that well approximates the original system at each local model configuration, and the frequency responses and eigenvalues of which evolve smoothly as the parameters vary.

A two step procedure is adopted to solve problem 1: (i) **local model reduction**, approximating the original local system model with a lower order one, while minimizing the mismatch error and preserving stability, and, (ii) **model interpolation** in order to construct a Linear Fractional Representation (LFR) on which control and analysis can be achieved. The rest of the paper is organized as follows: § "Approximation of large-scale LTI models" presents some literature approximation techniques and points out Onera's contributions. Then, interpolation issues and Onera's contributions to complete this step are presented in § "Algorithm for flexible aircraft LFT modeling". Then, § "Industrial application" presents very successful results from combining the reduction and interpolation phases, applied to an industrial aircraft model. Finally, we conclude and indicate directions for further development.

Approximation of large-scale LTI models

This section presents the model reduction step. This step is used to reduce the original model's complexity while keeping the model's main properties.

Preliminaries and problem formulation

There are two main families in the LTI model reduction field: (i) the projection based methods and (ii) the non-projection based ones. While the second methods are not well appropriated for a large-scale model [1, 20], the former methods clearly exhibit the best performances and will thus be considered in what follows. Mathematically, the problem considered in this section is given as in Problem 2.

Problem 2 - Projection-based linear dynamical systems approximation

Let us consider a MIMO dynamical model $G := (A, B, C)$ of an aircraft's dynamics at frozen configurations, defined by n-ODEs, as follows:

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n_u}, C \in \mathbb{R}^{n_y\times n}$

The projection-based model reduction problem consists of finding $V, W \in \mathbb{R}^{n\times r}$, where $W^T V = I_r$ ($r << n$), such that the reduced order model $\hat{G} := (\hat{A}, \hat{B}, \hat{C})$, defined as,

$$\hat{G} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t) \end{cases}$$

where $\hat{A} = W^T AV \in \mathbb{R}^{r\times r}, \hat{B} = W^T B \in \mathbb{R}^{r\times n_u}$ and $\hat{C} = CV \in \mathbb{R}^{n_u\times r}$, well approximates the original system, in the sense of a given metric.

Considering problem 2, the classical manner to assess **the quality of an approximation** is to consider the system's error by mean of a mathematical measurement. To do so, let us simply introduce the classical metrics widely used in the numerical and control communities, *i.e.* [1, 11]:

- The relative "mismatch error" metric (in %), based on the H_2 norm, defined as:

$$J_{H_2} = 100 \frac{\left\| H(s) - \hat{H}(s) \right\|_{H_2}}{\left\| H(s) \right\|_{H_2}}$$

where $\left\| H(j\omega) \right\|_{H_2} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} Tr\left(H(j\omega)^* H(j\omega) \right) d\omega}$

- The relative "worst case" error metric (in %), based on the H norm, defined as:

$$J_{H_\infty} = 100 \frac{\left\| H(s) - \hat{H}(s) \right\|_{H_\infty}}{\left\| H(s) \right\|_{H_\infty}}$$

where $\left\| H(j\omega) \right\|_{H_\infty} = \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left(H(j\omega) \right)$

In the model approximation framework, the objective is thus to reduce these errors. While the latter is practically very complex to achieve for large (medium) scale models due to the (iterative) nature of the H_∞ norm computation, when considering the former, first-order optimality conditions can be characterized and satisfied, practically, thanks to the celebrated Wilson conditions (see [13, 23, 24, 25, 26]).

(Non exhaustive) state of the art in the dynamical model approximation field

Methods that can be used to meet the Wilson first-order optimality conditions [25] have been widely explored and still are of great interest in both the numerical and control communities. Significant results in this field propose an iterative procedure for converging toward a near optimal condition. The underlying idea is to iteratively construct the projectors $V, W \in \mathbb{R}^{n\times r}$ using either the **Lyapunov and Sylvester-like** approaches [17, 26] or the **Tangential (Krylov)** ones [13, 23].

Tangent (Krylov) approaches

More specifically the following techniques, derived from the Tangential (Krylov)-like approaches, have retained a lot of attention in the recent years since they do obtain very nice results in practice and provide computational effectiveness:

- The Iterative Rational Krylov Algorithm (IRKA), initially set for SISO systems [13] which produces excellent results on benchmarks [7] but does not guarantee stability (unless implementing specific restart techniques). Later, in [3], the authors extended it to MIMO systems, with a complex Trust Region algorithm which guarantees convergence and preserves stability;
- At the same time, the Iterative Tangential Interpolation Algorithm

(ITIA) for MIMO systems, suggested in [23, 24], was developed to handle the MIMO case. Indeed, the ITIA, developed in [10, 23] is similar to the MIMO IRKA. Like the previous one, this procedure has proved to be effective on many classical benchmarks [7] but does not preserve stability, *a priori.*

The underlying idea of these methods is the moment matching. The moments are defined as follows.

Definition 1 - System moments

Let $H(s)=C(sI_n-A)^{-1}B$ be a complex valued MIMO rational transfer function. The system moments η_i, around the complex shift σ, are defined as a Laurent series given as follows,

$$\eta_i=-C(\sigma I_n-A)^{-(i+1)}B$$

and verify,

$$H(s)=\sum_{i=0}^{\infty}\eta_i(s-\sigma)^i$$

Because of the A matrix power, the moments computation is usually ill-conditioned and explicit moment matching is thus numerically impossible to achieve [20]. Consequently, the fact that the construction of Krylov subspaces allows for moment matching without computing them explicitly is used. Moreover, Krylov subspaces can be efficiently constructed through Arnoli-like procedures, very cheap from the computational point of view [1, 12, 20]. The main result of the moment matching, by construction of Krylov subspaces, is formulated in the following theorem (see *e.g.* [1, 13] and references therein).

Theorem 1 - Rational Krylov subspace and moment matching

Let $H(s)=C(sI_n-A)^{-1}B$ be a complex valued MIMO (n_u inputs, n_y outputs) rational transfer function, and $\sigma_k\in\mathbb{C}$ be k interpolation points such that (σ_kI-A) is invertible, if,

$$\begin{cases}\bigcup_{k=1}^{K}\mathbb{K}_r\left((\sigma_kI-A)^{-1},(\sigma_kI-A)^{-1}B\right)\subseteq span(V)\\ \bigcup_{k=1}^{K}\mathbb{K}_r\left((\sigma_kI-A)^{-*},(\sigma_kI-A)^{-*}C^T\right)\subseteq span(W)\end{cases}$$

where $\mathbb{K}_r(A,B)=\left[B,AB,...,A^{r-1}B\right]$ stands for the Krylov subspace of order r. Then, moments of the original and reduced models satisfy

$$\hat{\eta}_{\sigma_k}^{(j_k)}=\eta_{\sigma_k}^{(j_k)}\ \ for\ \ j_k=0,...,\left\lfloor\frac{r}{n_u}\right\rfloor+\left\lfloor\frac{r}{n_y}\right\rfloor-1$$

Based on this theorem, recent results within the numerical and control communities have provided proof and algorithms that can be used to meet the Wilson H_2 optimality conditions through the construction of iterative projectors (for more details, see [20, 25] and references therein).

Sylvester (and Lyapunov) like approaches

In parallel to the Tangent (Krylov)-like approaches, other techniques have been developed from the Sylvester and SVD approaches to approximate MIMO LTI systems (without always aiming at guaranteeing H_2 first-order optimality conditions), *e.g.*:

• Balanced Truncation (BT), which is often considered as the gold standard since it preserves stability, provides a bound on the error and a nearly optimal H_2 error. The drawback is that it may fail in practice when the system order is too large, because of the need to solve two Lyapunov equations [11, 27];

• The Low Rank Square Root Method (LRSRM), which is a modification of the BT approach, is applicable for large-scale models but does not guarantee the preservation of stability [17];

• Dominant Subspaces Projection Model Reduction (DSPMR), which is a heuristic approach that can be used to handle large-scale systems, without guaranteeing stability;

• The Two-Sided Iterative Algorithm (TSIA), which iteratively solves two Sylvester equations, has been shown to be equivalent to the tangential interpolation. This procedure guarantees stability and provides nice results for medium-scale problems but it suffers of two main drawbacks: first, it requires a good projector initialization to converge, and secondly, no stopping criterion has been described so far [26].

This second family of methods basically consists of solving either Lyapunov or Sylvester equations. As an illustration, Lyapunov-based approaches consist of solving the following equations:

$$\begin{cases}AP+PA^T+BB^T=0\\ A^TQ+QA+C^TC=0\\ P=Q=diag(\sigma_1,...,\sigma_n)\end{cases}$$

where $(\sigma_1,...,\sigma_n)$ are the matrix singular values

Then, states with high energy are kept, while the others are eliminated. When considering the Sylvester like approaches, the problem consists of solving a lower order equation of the form [26],

$$\begin{cases}AV+V\Sigma_\sigma+BR=0\\ W^TA+\Sigma_\mu^TW^T+L^TC=0\end{cases}$$

Many other methods exist, but the above ones catch our attention because of their efficiency. The main drawback on these approaches concerns the resolution of such equations.

Mixed approaches & Onera contribution

Nowadays, another family of methods is being increasingly explored: mixed ones. These methods combine the advantages of both methods. For deeper insight, readers are invited to refer to [12, 19]. Recently, Onera has made a contribution that is illustrated in Box 1.

Industrial aircraft application & comparison of methods

In this section, the model reduction techniques are applied to an industrial problem. The model considered is an industrial longitudinal aeroelastic model at varying flight points, as illustrated in figure 2 [20]. It is worth emphasizing that approximating (and controlling) such system is a challenging task since the model's order is about 300, the conditioning number is very high and numerous badly damped modes are present. The ISTIA approximation procedure is used on this industrial flexible aircraft model, and benchmarked with respect to the ITIA and the BT methods (note that BT is the one implemented in very efficient commercial computing software). In figure 3, the ε_{H_2} error (mismatch error) of models approximated with the BT, ITIA and ISTIA are plotted as a function of the approximation order r for a model at one single flight point.

Box 1 - Onera contribution (ISTIA) and tool developments

Based on [12] and [23], in [19], a new hybrid methodology has been proposed to allow for accurate LTI model approximation while preserving system stability. In the **MORE Toolbox**[1] very recent methods in the field of large-scale systems approximation, extracted both from the literature [12, 13 26] and from the project carried out within the laboratory [19, 20], have been implemented. One contribution is the definition and the numerical implementation of the SVD Tangential Interpolation Algorithm (ISTIA) [19], summarized as follows:

Algorithm ISTIA: Iterative SVD-Tangential Interpolation Algorithm [19]

Require: $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n_u}, C \in \mathbb{R}^{n_y \times n}, \left[\sigma_1^{(0)},\ldots,\sigma_r^{(0)}\right] \in \mathbb{C}^r, \left[\hat{b}_1,\ldots,\hat{b}_r\right] \in \mathbb{C}^{n\times r}, \varepsilon > 0$

1 Construct $V = span\left((\sigma_1^{(0)} I - A)^{-1} B\hat{b}_1,\ldots,(\sigma_r^{(0)} I - A)^{-1} B\hat{b}_r\right)$

2 Solve $A^T Q + QA + C^T C = 0$ in Q

3 Compute $W = QV(V^T QV)^{-1}$

4 While $\left|\sigma^{(i)} - \sigma^{(i-1)}\right| > \varepsilon$ Do

5 $i \leftarrow i+1$ and $\hat{A} = W^T AV, \hat{B} = W^T B$

6 Compute $\hat{A}X = diag\left(\lambda(\hat{A})\right)X$

7 Compute $\left[\hat{b}_1,\ldots,\hat{b}_r\right] = X^{-1}\hat{B}$

8 Set $\sigma^{(i)} = -\lambda(\hat{A})$

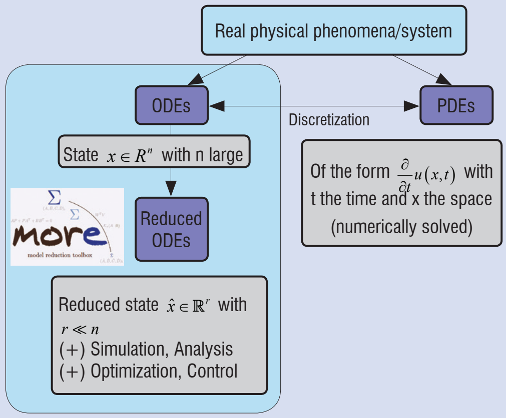
9 Construct $V = span\left((\sigma_1^{(i)} I - A)^{-1} B\hat{b}_1,\ldots,(\sigma_r^{(i)} I - A)^{-1} B\hat{b}_r\right)$

10 Compute $W = QV(V^T QV)^{-1}$

11 EndWhile

12 Construct $\hat{\Sigma} := (W^T AV, W^T B, CV)$

Ensure: $\hat{\Sigma} := (W^T AV, W^T B, CV)$ stable and partial H_2 optimality conditions



This algorithm has very nice theoretical and practical properties, such as, an almost H_2 optimal model approximant of the original one, while preserving stability at each step. The stopping criterion allows limiting the accuracy of the optimality criteria. Practically, the parameter is chosen small (*e.g.* 10^{-2}). It has been successfully applied on many large-scale models and on industrial flexible aircraft models, showing enhanced performances with respect to the classical techniques [19, 21]. On the following figure B1-1, the algorithm evolution is illustrated as it iterates, showing the mismatch error decrease and the interpolation points selection. Video - http://www.aerospacelab-journal.org/al4/scale-model-reduction-and-interpolation-with-application-to-aircraft-systems

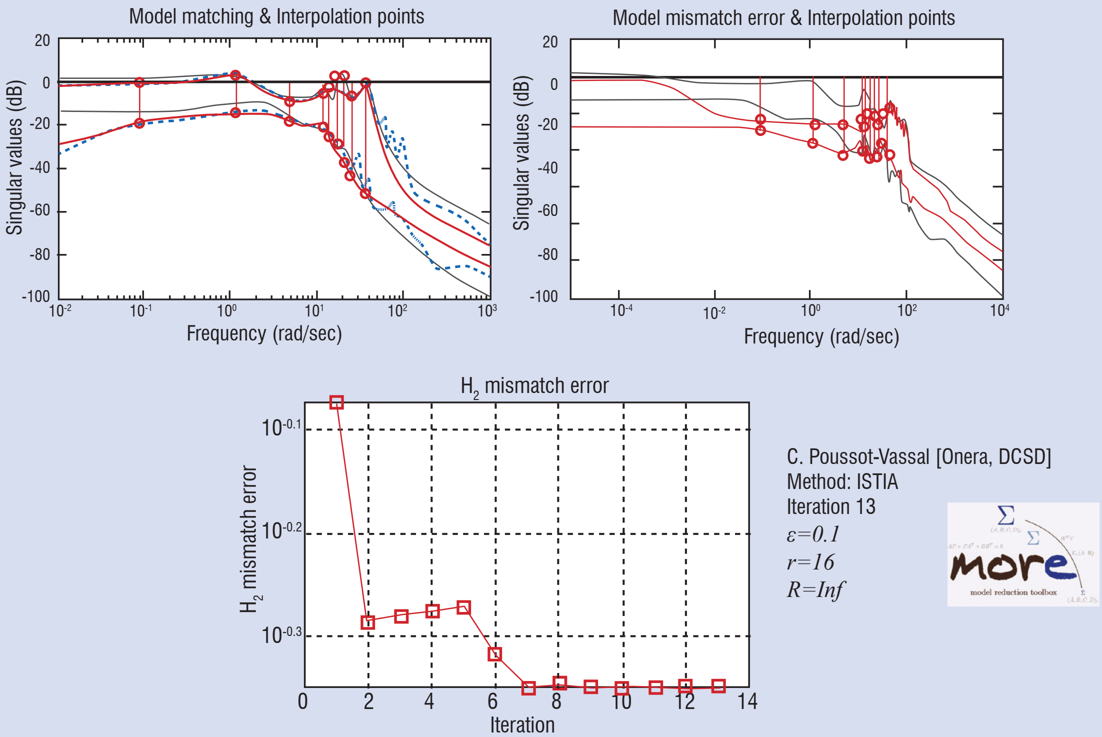


Figure B1-1 – Illustration of the algorithm evolution. Top left: frequency response (original, dashed blue / reduced, solid red). Top right: mismatch relative error (initial, solid black / reduced, solid red). Bottom left: mismatch error ε_{H_2} as a function of the iteration i.

[1] The MORE Toolbox - stands for MOdel REduction Toolbox (http://www.onera.fr/staff-en/charles-poussot-vassal) - is a dedicated medium(large)-scale LTI dynamical model approximation toolbox, developed within the Onera DCSD, by C. Poussot-Vassal.

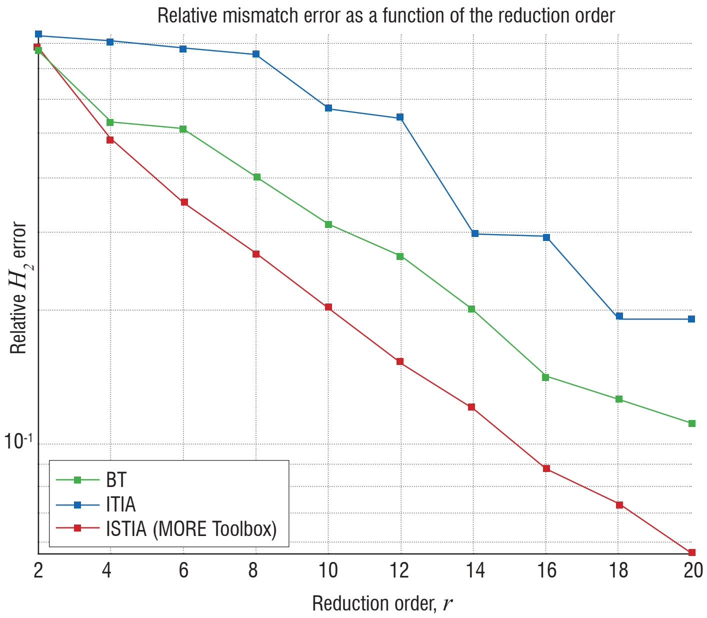


Figure 3 – Mismatch relative error (ε_{H_2}) as a function of the reduction order r, for one single flight configuration.

With reference to figure 3 it appears that the proposed ISTIA method outperforms the ITIA and BT approaches in terms of error mismatch in all situations. Next, figure 4 compares the frequency responses (left) and the eigenvalues locations (right) between the original and reduced models, with order 20, obtained with the ISTIA technique. Looking at this figure, it is clear that a good fit in terms of frequency response and pole location is achieved. This last point is crucial for engineers who are familiar with the physical meaning of model modes. Indeed, this specific feature is one of the advantages of the interpolation-based techniques, because they can focus on specific behaviors through the choice of initial interpolation points.

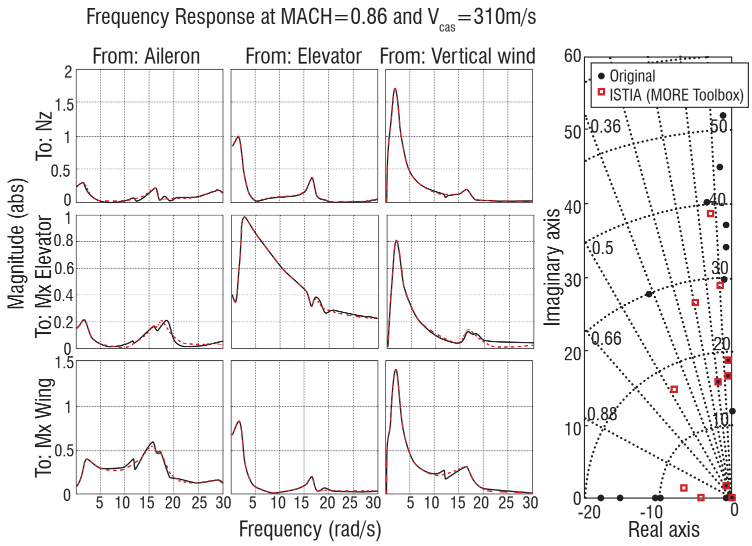


Figure 4 – Left: frequency response comparison of the original of order 286 (solid black) and approximated model of order 20 with ISTIA (dashed red). Right: eigenvalues of the original (black rounds) and approximated model with ISTIA (solid squares)

By performing this local reduction for every local LTI model, a collection of reduced order models are produced which can then (almost) be interpolated in order to build a parameterized model, describing the aircraft's behavior over the entire flight domain.

Algorithm for flexible aircraft LFT modeling

Based on the reduced order models, interpolation is now performed in order to generate an LFR model.

State coordinate transformation for state vector consistency

At this stage, the N reduced order models are available with the same number of modes which are not always of the same nature.

Now before interpolating the state space matrices, a state basis must be found that ensures that these matrices are consistent in terms of states whatever the flight point index i. More precisely, after the state matrices have been interpolated, the result must be regular modal trajectories as well as variations in frequency responses with respect to the flight parameters vector δ (see figure 5). This is an efficient test for state vector consistency.

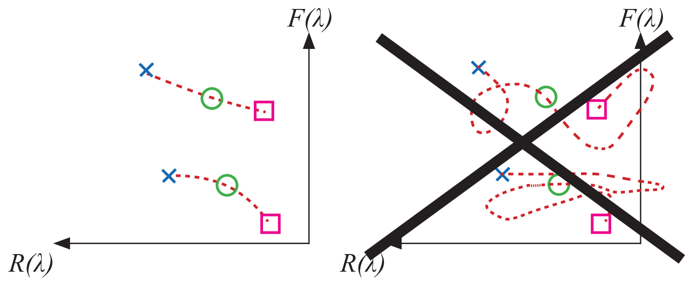


Figure 5 – Strong constraint: modal trajectories regularity

Our research showed that the characteristic polynomial of the models (1) is of deep interest as regards the modal trajectories regularity constraint. This phenomenon can be explained by the physical nature of this polynomial's coefficients (1) that are directly linked to the transfer function, and even more directly with the physical differential equations of the flexible aircraft.

$$d(s) = det(sI - A)$$
$$d(s) = s^{n_z} + c_{n_z-1}s^{n_z-1} + \cdots + c_1 s + c_0 \tag{1}$$

The state basis linked with the characteristic polynomial is the companion basis, in which the A matrix has the form:

$$A_i^{comp} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -c_0^i & -c_1^i & \cdots & \cdots & -c_{n_z-1}^i \end{pmatrix}$$

This companion state basis is known to provide badly-conditioned state matrices. So the A_i^{comp} matrices must be regularized via a scaling matrix T to balance the coefficients' values, while keeping the same eigenvalues [29]. The same scaling is applied to all models (*i.e.* $\forall i$ for consistency); a regularized companion matrix A_i^{compr} is then obtained:

$$A_i^{compr} = T^{-1} A_i^{comp} T$$

The scaling matrix T is computed for matrix A_i^{compr} so that it has the same rows and columns norms, as far as possible. More precisely, T is a diagonal matrix assembles integer powers of two on its diagonal, to avoid round off errors:

$$T = diag(2^{k_1}, \ldots, 2^{k_{n_z}})$$

in which $(k_p)_{p\in[1,n_x]}$ are the aforementioned integers, independent of the i index (flight point index).

The final regularized companion matrix A_i^{compr} is then:

$$A_i^{compr}=\begin{pmatrix} 0 & 2^{k_2-k_1} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{k_{n_z}-k_{n_z-1}} \\ -cr_0^i & -cr_1^i & \cdots & -cr_{n_z-1}^i \end{pmatrix} \quad (2)$$

where cr are the regularized characteristic polynomial coefficients.

Now the state space matrices corresponding to this regularized companion form (2) must be computed. To do so, the key step consists of linking the A_{compr} matrix with the initial A matrix by resorting to its modal form. Indeed, both matrices have one feature in common: their eigenvalues.

So, first the *(A,B,C)* model is expressed in the modal basis then the basis change matrix P_{mod} is computed so that:

$$\begin{cases} A_{mod}=P_{mod}^{-1}AP_{mod} \\ B_{mod}=P_{mod}^{-1}B \\ C_{mod}=CP_{mod} \end{cases}$$

The P_{mod} matrix has the form:

$$P_{mod}=P_{modc}P_{cr}=P_{modc}\begin{pmatrix} 1 & j & 0 & 0 & 0 \\ 1 & j & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & -j \\ 0 & 0 & 0 & 1 & j \end{pmatrix}$$

in which P_{modc} actually assembles the eigenvectors of A. Then, the same process is applied to A_{compr} matrix to get its modal form:

$$A_{mod}=P_{mod2}^{-1}A_{compr}P_{mod2}$$

The P_{mod2} basis change matrix has the same form as P_{mod} (its expression holds the eigenvector matrix of A_{compr} this time). For both basis changes, it can be shown that their generic expressions are:

$$P_{mod}=P_{modc}T_{\Delta}P_{cr}$$
$$P_{mod_2}=P_{modc_2}T_{\Delta_2}P_{cr}$$

where T_{Δ} and $T_{\Delta 2}$ are free (diagonal) scaling providing additional degrees of freedom and P_{modc} and P_{modc2} are matrices of eigenvectors of A. The latter can be used to help with the forthcoming state matrices interpolation (to improve the companion state matrices numerical conditioning or minimize their variations from one flight point to another).

Finally, the previous steps are summed up to compute the final basis change P such that:

$$\begin{cases} A_{compr}=P^{-1}AP \\ B_{compr}=P^{-1}B \\ C_{compr}=CP \end{cases} \quad (3)$$

We have:

$$\begin{aligned} A^{mod} &= P_{mod}^{-1}AP_{mod} &&= \left(P_{modc}T_{\Delta}P_{cr}\right)^{-1}AP_{modc}T_{\Delta}P_{cr} \\ &= P_{mod_2}^{-1}A^{compr}P_{mod_2} &&= \left(P_{modc2}T_{\Delta_2}P_{cr}\right)^{-1}A^{compr}P_{modc2}T_{\Delta_2}P_{cr} \end{aligned}$$

from which the A_{compr} matrix can be expressed with respect to the A matrix :

$$A^{compr}=\left(P_{modc2}T_{\Delta_2}P_{cr}\right)\left(P_{modc}T_{\Delta}P_{cr}\right)^{-1}A\left(P_{modc}T_{\Delta}P_{cr}\right)\left(P_{modc2}T_{\Delta_2}P_{cr}\right)^{-1} \quad (4)$$

Through identification using equation (3) and (4) the final basis change matrix is obtained:

$$P=P_{modc}T_{\Delta}T_{\Delta_2}^{-1}P_{modc2}^{-1}$$

Models interpolation and LFT modeling

Interpolation

The state space matrices are interpolated in their regularized companion form, through a multivariate polynomial structure $(p_j(\delta))_{j\in[1,N_t]}$. This problem can be easily solved with a least squares algorithm.

LFT realization

Once the interpolation structure is known, the LFT is simply obtained with the generalized Morton's method [16] that is implanted in the LFR toolbox (function **gmorton.m** [15]). This method is the generalization of the Morton's method to a polynomial expansion, and it relies on a singular value decomposition of each matrix coefficient.

Validation of the LFT

In order to assess the LFT accuracy, three criteria are defined: one evaluates the LFT modal matching with the reference models (5), and the other two are the $H\infty$ (6) and H_2 (7) frequential criteria for the frequency matching assessment.

$$\varepsilon_{modal}=\max_{i\in[1,N]}\left(\frac{\sum_{k=1}^{n_z}\left|\lambda_k^i-\lambda_k^{ref,i}\right|}{\sum_{k=1}^{n_z}\left|\lambda_k^{ref,i}\right|}\right) \quad (5)$$

where λ_k^i is the LFT's k-th mode at flight point number i and $\lambda_k^{ref,i}$ refers to the corresponding reference model $G_i(s)$.

$$\varepsilon_{H\infty}=\max_{i\in[1,N]}\left(\frac{\bar{\sigma}\left(\Delta F_i\left(jw\right)\right)}{\bar{\sigma}\left(G_i\left(jw\right)\right)}\right) \quad (6)$$

in which $\Delta F_i(j\omega)=(F_u(M(j\omega),\Delta^i)-G_i(j\omega))$ and $\bar{\sigma}$ is the maximum singular value on the pulsation continuum (H_∞ norm).

$$\varepsilon_{H_2}=\max_{i\in[1,N]}\left(\sqrt{\frac{1}{2\pi}\frac{\sum_j trace\left(\Delta F_i^*\left(jw_j\right)\Delta F_i\left(jw_j\right)\right)\Delta w_j}{trace\left(G_i^*\left(jw_j\right)G_i\left(jw_j\right)\right)}}\right) \quad (7)$$

where $\Delta\omega_j=(\omega_{j+1}-\omega_j)$

In depth validation is of course necessary to check both modal and frequential behaviors of the LFT on the whole model continuum. This step will be illustrated in the applicative example.

Input/Output error minimization

If the I/O error is not satisfactory, it can be minimized with a biconvex optimization. This algorithm is an extension to the LFT case of the one previously mentioned in paragraph 2. In this situation, the minimized criterion depends on the frequency error between the LFT and the reference models:

$$\Delta F_i(jw) = \left(F_u\left(M(jw), \Delta^i\right) - G_i(jw)\right)$$

Let us recall the state representation of an LFT:

$$\begin{cases} x = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases}$$

The LFT frequency response is then (the model index i is dropped for simplicity):

$$F_u\left(M(jw), \Delta\right) = C(\Delta) Y(jw, \Delta) B(\Delta) + D(\Delta)$$

with $Y(jw,\Delta) = \left(jwI - A(\Delta)\right)^{-1}$, $A(\Delta) = A + B_1 X_\Delta C_1$,

$B(\Delta) = B_2 + B_1 X_\Delta D_{12}$, $C(\Delta) = C_2 + D_{21} X_\Delta C_1$,

$D(\Delta) = D_{22} + D_{21} X_\Delta D_{12}$, $X_\Delta = \Delta\left(I - D_{11}\Delta\right)^{-1}$

Hence the two expressions of the frequency error are:

$$\Delta F(jw) = H_{BD}\begin{pmatrix} B_2 \\ D_{12} \\ D_{22} \end{pmatrix} - G(jw) = \left(c_2 D_{21} D_{22}\right) H_{CD} - G(jw)$$

with $H_{BD}^T(jw) = \begin{pmatrix} Y^T(jw,\Delta) C^T(\Delta) \\ X_\Delta^T\left[B_1^T Y^T(jw,\Delta) C^T(\Delta) + D_{21}^T\right] \\ I \end{pmatrix}$

and $H_{CD}(jw) = \begin{pmatrix} Y(jw,\Delta) B(\Delta) \\ X_\Delta\left[C_1 Y(jw,\Delta) B(\Delta) + D_{12}\right] \\ I \end{pmatrix}$

Back with the models indices *i.e.* i, each term of both H_2 criteria to be minimized:

$$J_{\Delta F \bullet \Delta F} = \sum_i \sum_i trace\left(\Delta F_i^*(jw_j) \Delta F_i(jw_j)\right)\left(w_{j+1} - w_j\right)$$

$$J_{\Delta F \Delta F \bullet} = \sum_i \sum_i trace\left(\Delta F_i(jw_j) \Delta F_i^*(jw_j)\right)\left(w_{j+1} - w_j\right)$$

has the following quadratic structure:

$$trace\left(\Delta F_i(jw_j) \Delta F_i^*(jw_j)\right) = c_{i,j} - 2\theta^T f_{i,j} + \theta^T Q_{i,j} \theta$$

where θ is a column vector obtained by concatenating either the columns of $\begin{pmatrix} B_2 \\ D_{12} \\ D_{22} \end{pmatrix}$ or the transpose of rows of $\left(C_2 D_{21} D_{22}\right)$.

The final expression is a quadratic criterion $c - 2\theta f + \theta^T Q\theta$ with $c = \sum_i \sum_j c_{i,j}(\omega_{j+1} - \omega_j)$, $f = \sum_i \sum_j f_{i,j}(\omega_{j+1} - \omega_j)$ and $Q = \sum_i \sum_j Q_{i,j}(\omega_{j+1} - \omega_j)$.

There is an analytical minimum at $\theta = Q^\bullet f$ ($\bullet$ is the Moore-Penrose pseudo inverse), which makes each loop of the biconvex optimization very fast.

Industrial application

As already mentioned in a paragraph above, this application illustrates the previously presented method of LFT modeling from a set of numerical models corresponding to a set of flight points and mass cases. These models are aircraft LFT longitudinal and lateral flexibilities for control design.

Description of the model

The set of aircraft models $(G_i(s))_{i\in[1,N]}$ correspond to variations of the parameters $\delta = \begin{pmatrix} OT & Ma & V_c \end{pmatrix}$, being respectively the outer tanks filling rate, Mach number and conventional airspeed.

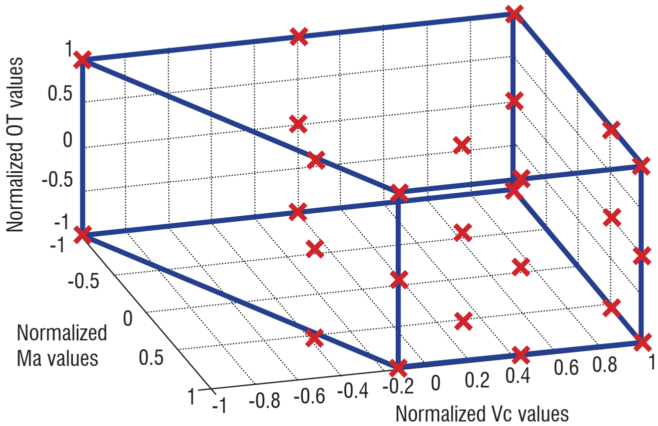


Figure 6 – Parametric domain of LFT model representativity . x : reference points for LFT modeling; - : parametric domain

These parameters vary inside the domain depicted in figure 6. For interpolation, $N = 27$ points are chosen inside this parametric domain.

The inputs of the model are the elevator δq and the ailerons in symmetric mode $\delta p_{sym} = \delta p_{left} + \delta p_{right}$.

The considered outputs are wing root bending load WRMX and wing root twisting load WRMY.

LFT construction

The LFT is then built according to the method presented in section 3.1, 3.2.1 and 3.2.2. The polynomial terms used for interpolation are computed by expanding the polynomial $(1+OT)^2(1+Ma)^2(1+Vc)^2$. This parameterization is sufficiently rich to obtain a very accurate interpolation. The obtained LFT has the following Δ-block:

$$\Delta = diag\left(\overline{OT} \times [\![_{42}, \overline{Ma} \times [\![_{30}, \overline{V_c} \times [\![_{52}\right)$$

$$dim(\Delta) = 110$$

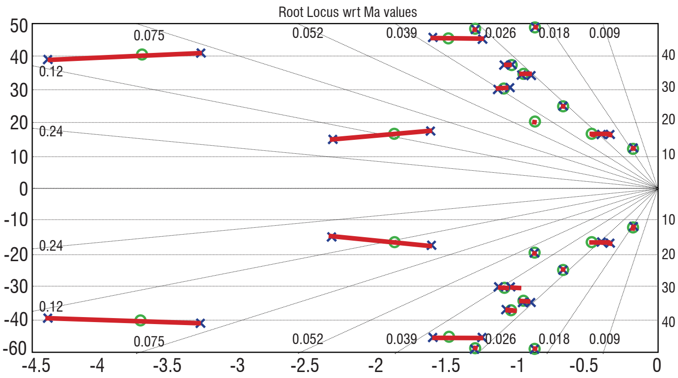


Figure 7 – Modal trajectories with respect to OT parameter variation
x: reference models modes; o: central model modes

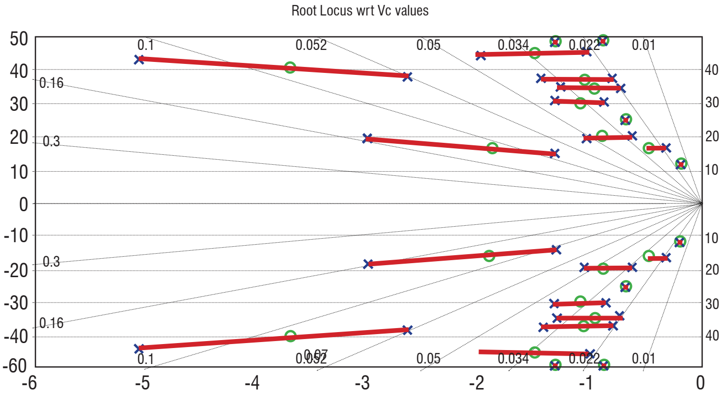


Figure 8 – Modal trajectories with respect to Vc parameter variation

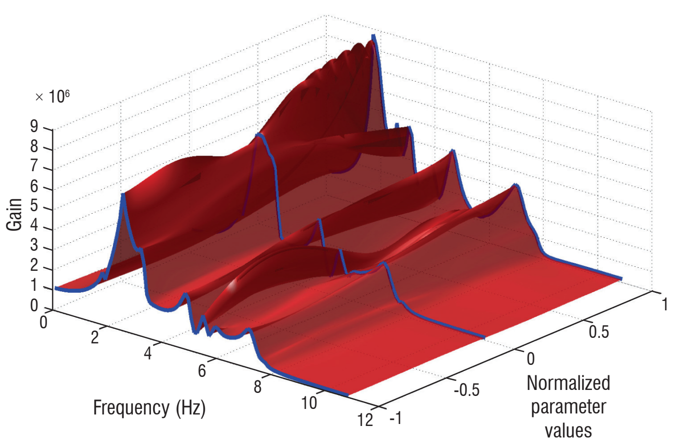


Figure 9 – Frequency response continuum with respect to Ma parameter
Solid lines: reference model's frequency responses

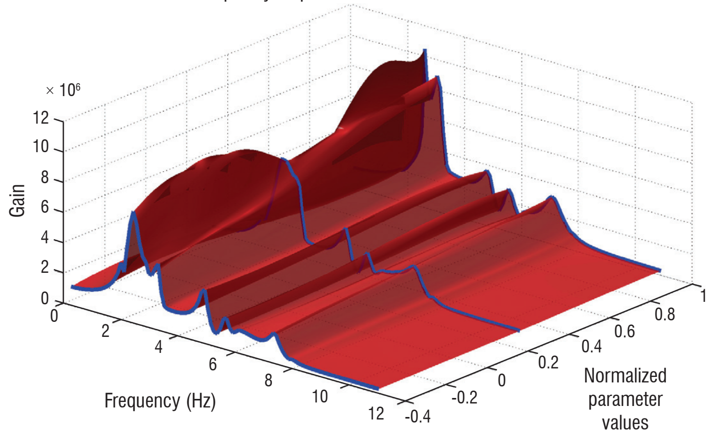


Figure 10 – Frequency response continuum with respect to Vc parameter
Solid lines: reference model's frequency responses

Validation of the LFT

The values of the validation criteria (5), (6) and (7) for the example are shown in table 1.

ε_{modal} %	$\varepsilon_{H\infty}$ %	ε_{H_2} %
1.27×10^{-5}	1.5×10^{-3}	9.28×10^{-4}

Table 1 – LFT Validation

Regularity check-up

Since an LFT is a continuum of models, the previously built LFT has also to be checked-up between the flight points used to design it. It must be proven that the continuum of modes (*i.e.* modal trajectories when δ varies) and the frequency response continuum are both regular. No "overshoot" must be observed, and ideally the continuum should vary linearly between two reference flight points.

In the application, the main directions of the parametric domain are explored to assess the regularity properties of the LFT.

The modal trajectories (see figure 7 and figure 8) show that the LFT has no unexpected behavior (*i.e.* no irregularities) in terms of modes. Besides, this proves the interest of the characteristic polynomial coefficients for A matrix interpolability. The frequency response continuum (figure 9 and figure 10) is fully satisfactory as well.

Conclusion

The method presented in this paper is used to design an LFT from a set of large-scale aeroelastic dynamical models. It is definitely adapted to complex and prominently numerical models, with no parametric structure knowledge whatsoever.

Naturally the least squares algorithm is used to interpolate the models with a basis of polynomials. Before interpolation, two steps are fateful in the process: the consistent reduction of the models and their state representations' transformation in a regularized companion state basis. In this way, the reduced models are made interpolable. After the LFT is created using the generalized Morton's method, its Input / Output accuracy can be optimized with an efficient biconvex optimization of the LFT state matrices. This algorithm was applied to both longitudinal and lateral aeroelastic models; the results showed very satisfactory modal trajectories and variations in frequency responses with about 20 states in both cases. This study, based on industrial complex aeroelastic models, clearly emphasized the efficiency of the tools provided by Onera.

These LFT models are well-adapted to full flight domain flexible aircraft control design. So these flexible LFT models are being used in the framework of research on the promising multi-objective flexible aircraft control extended to the full flight domain case [22] ■

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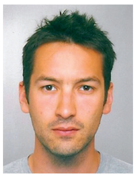
Acronyms

MIMO (Multiple Input Multiple Output)
SISO (Single Input Single Output)
LPV (Linear Parameter Varying)
LTI (Linear Time Invariant)
LMI (Linear Matrix Inequality)
MPC (Model Predictive Control)
ODE(s) (Ordinary Differential Equation(s))
PDE(s) (Partial Differential Equation(s))
$\bar{\sigma}(G)$ (Highest singular value)
iff. (if and only if)
resp. (respectively)



Charles Poussot-Vassal is researcher with the Onera, Flight Dynamics and Control System Department (DCSD). In 2005, he completed his Engineering degree and M.Sc. in Control and Embedded Systems from Grenoble INP-ESISAR and Lund University of Technology, respectively. In 2008, he completed his Ph.D. degree in Control Systems Theory, within the Grenoble INP. His main interests concern control system design, model reduction techniques and dynamical performance analysis, with applications in ground vehicles, web servers and aircraft systems.

Fabrice Demourant holds a PhD in Automatic Control from the National Aerospace College SUPAERO at Toulouse and was in charge of the development of control laws and analysis tools for aeronautic and space systems for 7 years. His main activities have been: control law design and modeling for large civil aircraft for Airbus and for blended wing body aircraft in the European ACFA2020 project (FP7), control law design for an in-house Onera project which concerns the satellites flying formation for along-track formation flight in low earth orbit, and robustness analysis of large size aeroelastic models and non linear and time-varying systems in the European NICE project supported by the European Defense Agency.